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The Two-Dimensional Two-Component Plasma Plus Background on a Sphere: Exact Results

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An exact solution is given for a two-dimensional model of a Coulomb gas, more general than the previously solved ones. The system is made up of a uniformly charged background, positive particles, and negative particles, on the surface of a sphere. At the special value $\Gamma=2$ of the reduced inverse temperature, the classical equilibrium statistical mechanics is worked out: the correlations and the grand potential are calculated. The thermodynamic limit is taken, and as it is approached the grand potential exhibits a finite-size correction of the expected universal form.

KEY WORDS: Coulomb gas; solvable models; finite-size corrections.

1. INTRODUCTION

At the special coupling $\Gamma=q^2/k_B T=2$, where q denotes the magnitude of the charges, T the temperature, and k_B Boltzmann's constant, the classical two-dimensional one-component and two-component plasma systems of point particles can be solved exactly in a number of different geometries. In the one-component case[1-3] the calculation is based on the Vandermonde determinant identity and is performed in the canonical ensemble, while in the two-component case[4-7] it is based on the Cauchy determinant identity and is performed in the grand canonical ensemble. Also, in the one-component case the classical Boltzmann factor at $\Gamma=2$ is isomorphic[8] to a

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squared wave function for N noninteracting fermions in two dimensions, subject to a uniform magnetic field and confined to the lowest Landau level. The grand partition function of the two-component plasma at $\Gamma=2$ is isomorphic[5] to the generating functional of the two-dimensional free Fermi field.

In Sections 2 and 3 we unify the exact calculation of the particle correlations for one-component[2] and two-component[7] plasma systems on a sphere. This is achieved by introducing a new formalism based on a determinant identity[9] which interpolates between the Vandermonde and Cauchy identities. In fact the formalism is of general applicability in the sense that it is not restricted to the sphere domain. The exact calculation is performed in an ensemble with a uniform background charge density of given total charge $-Nq$, a given number N of positive particles, and in addition a variable number M of pairs of positive and negative particles which is controlled by a fugacity ζ. In the limit $\zeta \to 0$ the canonical ensemble of the one-component plasma is reclaimed, while in the limit $N \to 0$ it is the grand canonical ensemble of the two-component plasma which is reclaimed.

In the infinite-plane system the correlations for the two-component plasma plus background system at $\Gamma=2$ are already known from the work of Cornu and Jancovici,[10] who used the formalism for the two-component plasma in a field[5] to perform the calculation. This suggests an alternative approach, considered in Section 4, to the system on a sphere: at the north pole fix a point charge Nq and then use the formalism of ref. 5. Of course the north pole becomes highly singular, as we expect that N particles of negative charge will accumulate about this point. However, due to the screening of the charge–charge correlations, we would expect this not to affect the correlations between charges away from the north pole. In fact, we find that this method reproduces the exact correlations of Section 3.

The grand potential Ω_R can be calculated by integration with respect to ζ of the expression for the density of negative charges. The large-R asymptotic expansion of Ω_R is of particular interest, as for a Coulomb system on a sphere it has been predicted that[6]

$$\Omega_R - \Omega_\infty \sim \frac{k_B T}{3} \ln R \tag{1.1}$$

independent of the details of the plasma. In Section 5, (1.1) is verified.

In the Appendix, it is shown how the partition function for the one-component plasma on a sphere can be calculated within the formalism introduced in Section 2.

2. FORMALISM BASED ON A DETERMINANT IDENTITY WHICH INTERPOLATES BETWEEN THE VANDERMONDE AND CAUCHY IDENTITIES

In this section the generalized (with position-dependent fugacities) grand canonical partition function for the two-component plasma plus background system at $\Gamma = 2$ will be expressed as the determinant of an integral operator. This allows the one-component and two-component plasma systems to be solved using the same technique. We will assume that the domain is the surface of a sphere for calculating the particle correlations in Section 3. However, the formalism could also be carried through in the other domains for which the one- and two-component plasmas are solvable, in particular the disk and semiperiodic boundary conditions.

On the surface of a sphere of radius R, the electric potential created by a particle of charge q at the angular distance ψ from it is[2]

$$\phi(\psi) = -q \ln\{[2R \sin(\psi/2)]/l\} \tag{2.1}$$

where l is an arbitrary length scale (to be taken as unity) and $2R \sin(\psi/2)$ is equal to the distance from the particle along the chord. This potential is indeed the Coulomb potential on a sphere, in the sense that, for a globally neutral system, the charges generate a total electric potential which obeys the Poisson equation: the spherical Laplacian of the potential equals -2π times the charge density.

Representing two particle positions by spherical polar coordinates θ and φ (θ is the angle from the north pole and φ is the usual other polar angle) and similarly θ' and φ', we find that their angular distance ψ obeys the identity[2]

$$\sin\left(\frac{\psi}{2}\right) = |\alpha\beta' - \alpha'\beta| = |\beta\beta'| \left|\frac{\alpha}{\beta} - \frac{\alpha'}{\beta'}\right| \tag{2.2a}$$

where

$$\alpha = \cos\frac{\theta}{2} e^{i\varphi/2}, \qquad \beta = -i \sin\frac{\theta}{2} e^{-i\varphi/2} \quad \text{(and similarly } \alpha', \beta'\text{)} \tag{2.2b}$$

The coordinates α and β are referred to as the Cayley–Klein parameters.

Consider now a system of $N+M$ particles of charge q with coordinates specified by Cayley–Klein parameters α'_j, β'_j ($j = 1,..., N+M$), M particles of charge $-q$ with coordinates specified by Cayley–Klein parameters α_j, β_j ($j = 1,..., M$), and a uniform neutralizing background. From (2.1) and (2.2), the full Boltzmann factor (including the constants from the background–particle and background–background interactions) is readily calculated as

$$W(N+M, N; \Gamma) = \left(\frac{1}{2R}\right)^{\Gamma\left(\frac{N}{2}+M\right)} e^{\Gamma N^2/4} \prod_{j=1}^{M} |\beta_j|^{-\Gamma(1+N)} \times \prod_{j=1}^{N+M} |\beta'_j|^{\Gamma(N-1)} |D|^{\Gamma} \tag{2.3a}$$

where

$$D := \frac{\prod_{1 \leqslant j<k \leqslant M} (u_k - u_j) \prod_{1 \leqslant j<k \leqslant N+M} (v_k - v_j)}{\prod_{j=1}^{M} \prod_{k=1}^{N+M} (u_j - v_k)} \tag{2.3b}$$

with

$$u_j := \frac{\alpha_j}{\beta_j} \quad \text{and} \quad v_j := \frac{\alpha'_j}{\beta'_j} \tag{2.3c}$$

Thus the generalized grand canonical partition function, with one-body potentials with Boltzmann factors $a(\theta', \varphi')$ and $b(\theta, \varphi)$ coupling to the positive and negative charges, respectively, and with N fixed and M summed over, is given by

$$\begin{aligned} &\Xi_\Gamma(a, b) \\ &= A_{N\Gamma} \sum_{M=0}^{\infty} \frac{1}{M!} \frac{\zeta^{2M}}{(N+M)!} \left(\frac{1}{2R}\right)^{\Gamma M} R^{4M} \\ &\quad\times \left[\prod_{l=1}^{N+M} \int_0^\pi d\theta'_l \sin\theta'_l \left(\sin\frac{\theta'_l}{2}\right)^{-\Gamma(1-N)} \int_0^{2\pi} d\varphi'_l\, a(\theta'_l, \varphi'_l)\right] \\ &\quad\times \left[\prod_{l=1}^{M} \int_0^\pi d\theta_l \sin\theta_l \left(\sin\frac{\theta_l}{2}\right)^{-\Gamma(1+N)} \int_0^{2\pi} d\varphi_l\, b(\theta_l, \varphi_l)\right] |D|^\Gamma \end{aligned} \tag{2.4a}$$

where

$$A_{N\Gamma} = R^{2N} \left(\frac{1}{2R}\right)^{N\Gamma/2} e^{\Gamma N^2/4} \tag{2.4b}$$

Actually, the integrals in (2.4a) are divergent, because particles of opposite sign have a tendency to collapse on each other, as described by the possible vanishing of the denominator in (2.3b). This divergence already occurred for the two-component plasma.[5] From now on, the divergences are to be understood as suppressed by some short-distance regularization [see Eqs. (5.3)].

In the case $N=0$ (two-component plasma) D can be written as a Cauchy determinant, while in the case $M=0$ (one-component plasma) D can be written as a Vandermonde determinant. In the general case we have[9]

$$D=(-1)^{M(M-1)/2}\det \mathbb{A} \tag{2.5a}$$

where

$$\mathbb{A}=\begin{bmatrix} 1 & \cdots\cdots & 1 \\ v_1 & \cdots\cdots & v_{M+N} \\ \vdots & & \vdots \\ v_1^{N-1} & \cdots\cdots & v_{M+N}^{N-1} \\ \dfrac{1}{u_1-v_1} & \cdots\cdots & \dfrac{1}{u_1-v_{M+N}} \\ \vdots & & \vdots \\ \dfrac{1}{u_M-v_1} & \cdots\cdots & \dfrac{1}{u_M-v_{M+N}} \end{bmatrix} \tag{2.5b}$$

and consequently

$$|D|^2=\det\begin{bmatrix} \mathbb{O} & i\mathbb{A} \\ i\mathbb{A}^+ & \mathbb{O} \end{bmatrix} \tag{2.6}$$

where $\mathbb{O}$ denotes the $(M+N)\times(M+N)$ zero matrix.

Consider now the generalized grand partition function (2.4a) for $\Gamma=2$ with the substitution (2.6). Suppose we discretize the integrals in (2.4a) by making the replacements

$$\begin{aligned} \int_0^\pi d\theta_l\, f(\theta_l;\ldots) &\to \frac{1}{K_1}\sum_{m_l=1}^{K_1} f\left(\frac{\pi(m_l-\nu_1)}{K_1};\ldots\right) \\ \int_0^\pi d\theta'_l\, f(\theta'_l;\ldots) &\to \frac{1}{K'_1}\sum_{m'_l=1}^{K'_1} f\left(\frac{\pi(m'_l-\nu'_1)}{K'_1};\ldots\right) \\ \int_0^{2\pi} d\varphi_l\, f(\varphi_l;\ldots) &\to \frac{1}{K_2}\sum_{n_l=1}^{K_2} f\left(\frac{2\pi(n_l-\nu_2)}{K_2};\ldots\right) \\ \int_0^{2\pi} d\varphi'_l\, f(\varphi'_l;\ldots) &\to \frac{1}{K'_2}\sum_{n'_l=1}^{K'_2} f\left(\frac{2\pi(n'_l-\nu'_2)}{K'_2};\ldots\right) \end{aligned} \tag{2.7}$$

where $0 < \nu_1, \nu'_1, \nu_2, \nu'_2 < 1$. The domain available to the particles now consists of separate sublattices for the positive and negative charges ($\nu_1, ..., \nu'_2$ are to be chosen so that no two lattice points overlap). We impose the constraint that the number of lattice points available to the positive charges, $K'_1 K'_2$, is N greater than the number available to the negative charges, $K_1 K_2$. The replacements (2.7) are exact in the limit $K_1, ..., K'_2 \to \infty$, so we can write

$$\begin{aligned}\Xi_2[a, b] = A_{N2} &\lim_{K_1, K'_1, K_2, K'_2 \to \infty} \sum_{M=0}^{K_1 K_2} \frac{1}{M!\,(M+N)!} \left(\frac{\zeta R}{2}\right)^{2M} \\ &\times \prod_{l=1}^{M+N} \left\{ \frac{1}{K'_1} \sum_{m'_l=1}^{K'_1} \left(\sin \frac{\pi(m'_l - \nu'_1)}{K'_1}\right) \left(\sin \frac{\pi(m'_l - \nu'_1)}{2K'_1}\right)^{-2(1-N)} \right. \\ &\left. \times \frac{1}{K'_2} \sum_{n'_l=1}^{K'_2} a\left(\frac{\pi(m'_l - \nu'_1)}{K'_1}, \frac{2\pi(n'_l - \nu'_2)}{K'_2}\right) \right\} \\ &\times \prod_{l=1}^{M} \left\{ \frac{1}{K_1} \sum_{m_l=1}^{K_1} \left(\sin \frac{\pi(m_l - \nu_1)}{K_1}\right) \left(\sin \frac{\pi(m_l - \nu_1)}{2K_1}\right)^{-2(1+N)} \right. \\ &\left. \times \frac{1}{K_2} \sum_{n_l=1}^{K_2} b\left(\frac{\pi(m_l - \nu_1)}{K_1}, \frac{2\pi(n_l - \nu_2)}{K_2}\right) \right\} \det \begin{bmatrix} \mathbb{O} & i\mathbb{A} \\ i\mathbb{A}^+ & \mathbb{O} \end{bmatrix} \end{aligned} \tag{2.8}$$

where the coordinates implicit in $\mathbb{A}$ are taken as lattice points according to the prescription (2.7).

We now make the crucial observation that the expression after the limit in (2.8) is precisely the expanded form of a single determinant:

$$\Xi_2[a, b] = A_{N2} \lim_{K_1, K'_1, K_2, K'_2 \to \infty} \det \left(\mathbb{1}' + \begin{bmatrix} \mathbb{O} & i\mathbb{B} \\ i\mathbb{C} & \mathbb{O} \end{bmatrix} \right) \tag{2.9}$$

where $\mathbb{1}'$ is a $2K'_1 K'_2 \times 2K'_1 K'_2$ diagonal matrix with the first N diagonal entries zero and the rest one, $\mathbb{O}$ is the $K'_1 K'_2 \times K'_1 K'_2$ zero matrix, and

$$\mathbb{B} = \begin{bmatrix} \left[\dfrac{1}{K'_1 K'_2} v_{j'k'}^{p-1} \right]_{\substack{p=1,...,N \\ (j'k')=(11),...,(K'_1 K'_2)}} \\ \left[\dfrac{(\zeta R/2)\, b_{jk}}{K'_1 K'_2 (u_{jk} - v_{j'k'})} \right]_{\substack{(jk)=(11),...,(K_1 K_2) \\ (j'k')=(11),...,(K'_1 K'_2)}} \end{bmatrix} \tag{2.10a}$$

$$\mathbb{C} = \left[\left[\bar{v}_{j'k'}^{p-1} a_{j'k'} \right]_{\substack{(j'k')=(11),...,(K'_1 K'_2) \\ p=1,...,N}} \left[\frac{(\zeta R/2)\, a_{j'k'}}{K_1 K_2 (\bar{u}_{jk} - \bar{v}_{j'k'})} \right]_{\substack{(j'k')=(11),...,(K'_1 K'_2) \\ (jk)=(11),...,(K_1 K_2)}} \right] \tag{2.10b}$$

(the upper index on the matrix symbols $[\cdot]$ labels the rows, while the lower index labels the columns), where

$$b_{jk} = \sin[\pi(j-\nu_1)/K_1]\{\sin[\pi(j-\nu_1)/2K_1]\}^{-2(1+N)}$$
$$\times b(\pi(j-\nu_1)/K_1, 2\pi(k-\nu_2)/K_2) \tag{2.11a}$$
$$a_{j'k'} = \sin[\pi(j'-\nu'_1)/K'_1]\{\sin[\pi(j'-\nu'_1)/2K'_1]\}^{-2(1-N)}$$
$$\times a(\pi(j'-\nu'_1)/K'_1, 2\pi(k'-\nu'_2)/K'_1) \tag{2.11b}$$

and u_{jk} denotes the coordinate $u=\alpha/\beta$ with θ and φ therein evaluated at the lattice point $\theta=\pi(j-\nu_1)/K_1$, $\varphi=2\pi(k-\nu_2)/K_2$ (and similarly for the meaning of $v_{j'k'}$ in terms of $v=\alpha'/\beta'$).

From (2.10) and (2.11) we see that the limit in (2.9) can be formally taken to give

$$\Xi_2[a, b] = A_{N2}\det(\mathbb{X}) \tag{2.12}$$

where the operator $\mathbb{X}$ acts on vectors

$$\psi := \begin{bmatrix} a_1 \\ \vdots \\ a_N \\ f(\theta, \varphi) \\ g(\theta, \varphi) \end{bmatrix} \tag{2.13}$$

and is defined by

$$\mathbb{X}\psi := \begin{bmatrix} A_1 \\ \vdots \\ A_N \\ F(\theta, \varphi) \\ G(\theta, \varphi) \end{bmatrix} \tag{2.14}$$

where

$$A_j := i\int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\varphi\, [\sin(\theta/2)]^{N-1}$$
$$\times [\cot(\theta/2)\, e^{i\varphi}]^{j-1} g(\theta, \varphi) \tag{2.15a}$$

$$F(\theta,\varphi)=f(\theta,\varphi)-\frac{i\zeta R}{2}\int_0^\pi d\theta_1\sin\theta_1\left(\sin\frac{\theta_1}{2}\right)^{N-1}\left(\sin\frac{\theta}{2}\right)^{-1-N}$$

$$\times\int_0^{2\pi}d\varphi_1\frac{g(\theta_1,\varphi_1)\,b(\theta_1,\varphi_1)}{\cot(\theta_1/2)\,e^{i\varphi_1}-\cot(\theta/2)\,e^{i\varphi}} \tag{2.15b}$$

$$G(\theta,\varphi)=g(\theta,\varphi)+i\sum_{j=1}^{N}a_j\left[\cot\left(\frac{\theta}{2}\right)e^{-i\varphi}\right]^{j-1}\left(\sin\frac{\theta}{2}\right)^{N-1}a(\theta,\varphi)$$

$$+\frac{i\zeta R}{2}\int_0^\pi d\theta_1\sin\theta_1\left(\sin\frac{\theta_1}{2}\right)^{-N-1}\left(\sin\frac{\theta}{2}\right)^{N-1}$$

$$\times\int_0^{2\pi}d\varphi_1\frac{f(\theta_1,\varphi_1)\,a(\theta_1,\varphi_1)}{\cot(\theta_1/2)\,e^{-i\varphi_1}-\cot(\theta/2)\,e^{-i\varphi}} \tag{2.15c}$$

In the case $N=0$, the generalized grand partition function (2.12) reduces to the known expression[5] for the grand partition function of the two-component plasma. In the other limiting case, $\zeta=0$, corresponding to the one-component plasma, the resulting expression for the generalized partition function is new; we show in the Appendix how to use (2.12) in this case to reclaim the result of ref. 2 for the exact evaluation of the partition function. In the general case $N\neq 0$, $\zeta\neq 0$, we have not been able to compute the grand potential directly from (2.12), but we shall obtain it from a density computed in the next section.

3. EVALUATION OF THE CORRELATION FUNCTIONS

The fully truncated n-particle distribution functions can be obtained by functional differentiation of the logarithm of (2.12). Analogous to the situation in the two-component plasma limit,[5] the distributions can be expressed in terms of functions which play the same role as the Green functions of ref. 5:

$$G_{s_1s_2}((\theta_1,\varphi_1),(\theta_2,\varphi_2)):=\delta_{s_1s_2}\delta((\theta_1,\varphi_1),(\theta_2,\varphi_2))$$
$$-A_{s_1s_2}((\theta_1,\varphi_1),(\theta_2,\varphi_2)) \tag{3.1}$$

where $s_1, s_2=\pm$, $\delta((\theta_1,\varphi_1),(\theta_2,\varphi_2))$ denotes the Dirac delta function on the surface of the sphere:

$$R^2\int_0^\pi d\theta\sin\theta\int_0^{2\pi}d\varphi\, H(\theta,\varphi)\,\delta((\theta,\varphi),(\theta',\varphi'))=H(\theta',\varphi') \tag{3.2}$$

for any continuous function $H(\theta, \varphi)$, and $A_{s_1 s_2}$ is defined in terms of $\mathbb{X}^{-1}$ by

$$\mathbb{X}^{-1} = \begin{bmatrix} \alpha_{11} \cdots\cdots \alpha_{1N} & F_1(\theta_2, \varphi_2) & G_1(\theta_2, \varphi_2) \\ \vdots \qquad \vdots & \vdots & \vdots \\ \alpha_{N1} \cdots\cdots \alpha_{NN} & F_N(\theta_2, \varphi_2) & G_N(\theta_2, \varphi_2) \\ f_1(\theta_1, \varphi_1) \cdots f_N(\theta_1, \varphi_1) & A_{++}((\theta_1, \varphi_1), (\theta_2, \varphi_2)) & A_{+-}((\theta_1, \varphi_1), (\theta_2, \varphi_2)) \\ g_1(\theta_1, \varphi_1) \cdots g_N(\theta_1, \varphi_1) & A_{-+}((\theta_1, \varphi_1), (\theta_2, \varphi_2)) & A_{--}((\theta_1, \varphi_1), (\theta_2, \varphi_2)) \end{bmatrix} \tag{3.3}$$

The one-body density of particles of sign s is

$$\rho_s(\theta, \varphi) = G_{ss}((\theta, \varphi), (\theta, \varphi)) \tag{3.4a}$$

the two-body truncated density is

$$\begin{aligned} &\rho^T_{s_1 s_2}((\theta_1, \varphi_1), (\theta_2, \varphi_2)) \\ &\quad = -G_{s_1 s_2}((\theta_1, \varphi_1), (\theta_2, \varphi_2))\, G_{s_2 s_1}((\theta_2, \varphi_2), (\theta_1, \varphi_1)) \end{aligned} \tag{3.4b}$$

and similarly for the fully truncated n-particle distribution.

Due to the rotational invariance of the correlations, it suffices to calculate $\rho^T_{s_1 s_2}$ with one of the particles fixed at the south pole ($\rho^T_{s_1 s_2}$ will then depend on θ only). Thus, from (3.4), it suffices to calculate

$$G_{s_1 s_2}((\theta, \varphi), (\pi, \varphi')) := G_{s_1 s_2}(\theta, \varphi) \tag{3.5}$$

From the definition (3.3) of the $A_{s_1 s_2}$ as elements of the inverse of the operator $\mathbb{X}$, the definition (2.14) of $\mathbb{X}$, and (3.1), it follows that the functions $G_{-+}(\theta, \varphi)$ and $G_{++}(\theta, \varphi)$ satisfy the coupled equations

$$\begin{aligned} &\int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi\, [\sin(\theta/2)]^{N-1}\, [\cot(\theta/2)\, e^{i\varphi}]^{j-1}\, G_{++}(\theta, \varphi) \\ &\quad = \begin{cases} 1/R^2, & j = 1 \\ 0, & j = 2, \dots, N \end{cases} \end{aligned} \tag{3.6a}$$

$$\begin{aligned} &G_{-+}(\theta_1, \varphi_1) + \frac{i\zeta R}{2} \int_0^\pi d\theta_2 \sin\theta_2 \left(\sin\frac{\theta_2}{2}\right)^{N-1} (\sin\theta_1)^{-1-N} \\ &\quad \times \int_0^{2\pi} d\varphi_2 \frac{G_{++}(\theta_2, \varphi_2)}{\cot(\theta_1/2)\, e^{i\varphi_1} - \cot(\theta_2/2)\, e^{i\varphi_2}} \\ &\quad = \frac{i\zeta}{2R} \frac{[\sin(\theta_1/2)]^{-1-N}}{\cot(\theta_1/2)} e^{-i\varphi_1} \end{aligned} \tag{3.6b}$$

$$-i \sum_{j=1}^{N} \left[\cot\left(\frac{\theta_1}{2}\right) e^{-i\varphi_1} \right]^{j-1} \left(\sin\frac{\theta_1}{2}\right)^{N-1} G_j(\pi)$$

$$+ \frac{i\zeta R}{2} \int_0^{\pi} d\theta_2 \sin\theta_2 \left(\sin\frac{\theta_2}{2}\right)^{-N-1} \left(\sin\frac{\theta_1}{2}\right)^{N-1}$$

$$\times \int_0^{2\pi} \frac{d\varphi_2\, G_{-+}(\theta_2, \varphi_2)}{\cot(\theta_2/2)\, e^{-i\varphi_2} - \cot(\theta_1/2)\, e^{-i\varphi_1}} + G_{++}(\theta_1, \varphi_1)$$

$$= 0 \tag{3.6c}$$

Also, $G_{+-}(\theta, \varphi)$ and $G_{--}(\theta, \varphi)$ satisfy the same coupled equations, with the replacements

$$G_{++}(\theta, \varphi) \to G_{+-}(\theta, \varphi), \qquad G_{-+}(\theta, \varphi) \to G_{--}(\theta, \varphi)$$

$$\text{r.h.s. of (3.6a)} \to 0 \quad \text{each } j = 1, \ldots, N$$

$$\text{r.h.s. of (3.6b)} \to 0 \tag{3.7}$$

$$\text{r.h.s. of (3.6c)} \to -\frac{i\zeta[\sin(\theta_1/2)]^{N-1}}{2R\cot(\theta_1/2)}\, e^{i\varphi_1}$$

Let us consider the coupled equations (3.6) for G_{-+} and G_{++}. We observe that these equations permit a solution of the form

$$G_{++}(\theta, \varphi) = \gamma_{++}(\theta) \tag{3.8a}$$

$$G_{-+}(\theta, \varphi) = \gamma_{-+}(\theta)\, e^{-i\varphi} \tag{3.8b}$$

$$G_j(\pi) = 0, \qquad j = 2, \ldots, N \tag{3.8c}$$

Substituting (3.8) in (3.6), changing variables $t = \cos\theta$ in the resulting equations, and setting

$$\gamma_{-+}(\theta) = \frac{h_{-+}(t)}{(1-t)^{N/2}(1+t)^{1/2}} \tag{3.9a}$$

$$\gamma_{++}(\theta) = \frac{h_{++}(t)}{(1-t)^{(N-1)/2}} \tag{3.9b}$$

then gives the coupled equations

$$2^{-(N-1)/2} \int_{-1}^{1} dt_2\, h_{++}(t_2) = \frac{1}{2\pi R^2} \tag{3.10a}$$

$$h_{-+}(t) + 2\pi i\zeta R \int_{-1}^{t} dt_2\, h_{++}(t_2) = \frac{i\zeta}{2R} 2^{(1+N)/2} \tag{3.10b}$$

$$-i \cdot 2^{-(N-1)/2} G_1(\pi) + 2\pi i\zeta R \int_t^1 dt_2 \frac{h_{-+}(t_2)}{(1+t_2)(1-t_2)^N} + (1-t)^{-(N-1)} h_{++}(t) = 0 \tag{3.10c}$$

Differentiating (3.10b) and (3.10c) with respect to t, substituting the first of the resulting equations in the second, and simplifying gives

$$(1-t^2)\, h''_{-+}(t) + (N-1)(1+t)\, h'_{-+}(t) - (2\pi\zeta R)^2\, h_{-+}(t) = 0 \tag{3.11}$$

This is the Gauss hypergeometric differential equation, in the variable

$$s = (1-t)/2 \tag{3.12}$$

with (in standard notation)

$$\begin{aligned} a &= -\tfrac{1}{2}[N - (N^2 - 4(2\pi\zeta R)^2)^{1/2}] \\ b &= -\tfrac{1}{2}[N + (N^2 - 4(2\pi\zeta R)^2)^{1/2}] \\ c &= 1 - N \end{aligned} \tag{3.13}$$

We note from (3.10c) that we require the solution of (3.11) such that

$$h_{-+}(s) = o(s^{N-1}) \qquad \text{as} \quad s \to 0 \tag{3.14}$$

[otherwise the intregral in (3.10c) does not exist; furthermore, (3.14) ensures that (3.10a) is satisfied]. The solution of (3.11) with this property is[11]

$$\begin{aligned} h_{-+}(s) &= A s^N F(a+N, b+N; N+1; s) \\ &= A s^N (1-s)\, F(1-a, 1-b; N+1; s) \end{aligned} \tag{3.15}$$

To determine A, we note from (3.10b) the requirement

$$h_{-+}(s=1) = \frac{i\zeta}{2R} 2^{1+(N/2)} \tag{3.16}$$

This gives

$$A = \frac{i\zeta}{2R} 2^{1+(N/2)} \frac{\Gamma(1-a)\, \Gamma(1-b)}{\Gamma(N+1)} \tag{3.17}$$

Now that h_{-+} is known, the value of h_{++} follows from differentiating (3.10b) with respect to s, which gives

$$h_{++}(s) = \frac{1}{4\pi i\zeta R} h'_{-+}(s)$$

$$= \frac{1}{4\pi i\zeta R} ANs^{N-1} F(a+N, b+N; N; s) \tag{3.18}$$

Reverting back to the original variable, using (3.12) with $t = \cos\theta$, and then to the original Green's function, via (3.9) and (3.8), we thus have

$$G_{-+}(\theta, \varphi) = e^{-i\varphi} \frac{i\zeta}{2R} \frac{\Gamma(1-a)\,\Gamma(1-b)}{\Gamma(N+1)} \cos\left(\frac{\theta}{2}\right) \sin^N\left(\frac{\theta}{2}\right)$$
$$\times F\left(1-a, 1-b; N+1; \sin^2\left(\frac{\theta}{2}\right)\right) \tag{3.19a}$$

and, after a simple manipulation on the Γ functions,

$$G_{++}(\theta, \varphi) = \pi\zeta^2 \frac{\Gamma(N+a)\,\Gamma(N+b)}{\Gamma(N)} \sin^{N-1}\left(\frac{\theta}{2}\right)$$
$$\times F\left(a+N, b+N; N; \sin^2\left(\frac{\theta}{2}\right)\right) \tag{3.19b}$$

To solve Eqs. (3.6) with the replacements (3.7), we try for a solution of the form

$$G_{--}(\theta, \varphi) = \gamma_{--}(\theta) \tag{3.20a}$$

$$G_{+-}(\theta, \varphi) = \gamma_{+-}(\theta)\, e^{i\varphi} \tag{3.20b}$$

$$F_j(\pi) = 0, \qquad j = 1,..., N \tag{3.20c}$$

Proceeding as in the calculation of G_{++} and G_{-+} above, we find

$$G_{+-}(\theta, \varphi) = \overline{G_{-+}(\theta, \varphi)} \tag{3.21a}$$

and

$$G_{--}(\theta, \varphi) = \pi\zeta^2 \frac{\Gamma(a+N+1)\,\Gamma(b+N+1)}{\Gamma(N+2)} \sin^{N+1}\left(\frac{\theta}{2}\right)$$
$$\times F\left(a+N+1, b+N+1; N+2; \sin^2\left(\frac{\theta}{2}\right)\right) \tag{3.21b}$$

4. ALTERNATIVE EVALUATION OF THE CORRELATION FUNCTIONS

In this approach, we put on the sphere a uniform background charge density of total charge $-Nq$ and a point charge Nq at the north pole, with N fixed. Then, we add a variable number $N+M$ of pairs of positive and negative particles. We expect that N negative particles will stick to the north pole, and therefore there will remain the background, $N+M$ positive mobile particles, and M negative mobile particles, as in the previous section.

Now, we can use the grand-canonical formalism, with the same variable number $N+M$ of positive and negative particles. The charge Nq at the north pole generates a one-body potential, which can be taken into account through a charge- and position-dependent fugacity. From (2.1), when $\Gamma := q^2/k_B T = 2$, this fugacity is of the form $\zeta[\sin^2(\theta/2)]^{sN}$ for a particle of sign s ($s = \pm 1$).

The problem on the sphere can be transformed into a problem in the plane by the same stereographic projection as in ref. 7. Let P be the stereographic projection of a point M of the sphere, from the north pole, onto the plane tangent to the south pole. In terms of the spherical coordinates (θ, φ) of M, the complex coordinate of P in the plane is

$$z = 2Re^{i\varphi} \cot \frac{\theta}{2} \tag{4.1}$$

The projection is a conformal transformation, with a conformal weight

$$e^{\omega} := \sin^2 \frac{\theta}{2} = \frac{1}{1 + (|z|^2/4R^2)} \tag{4.2}$$

That is, an element of length dl at M and its projection of length $|dz|$ at P have a ratio $dl/|dz| = e^{\omega}$. Also, in terms of the coordinates z and z' of the projections of the particles, the pair potential (2.1) becomes

$$\phi = -qq' \left(\ln |z - z'| + \frac{\omega}{2} + \frac{\omega'}{2} \right) \tag{4.3}$$

Since each particle interacts with $N+M-1$ particles of the same sign and $N+M$ particles of the opposite sign, the ω term in (4.3) gives a total one-body potential $q^2\omega/2$; for $\Gamma = 2$, this generates a factor $e^{-\omega}$ in the fugacity. Furthermore, in the computation of the partition function, the area element dS on the sphere can be expressed in terms of the corresponding area element d^2z on the plane as $dS = e^{2\omega} d^2z$. Altogether, the system on the sphere with a fugacity $\zeta e^{\omega s N}$ and the pair potential (2.1) is equivalent to a system in the plane with a fugacity

$$\zeta_s(|z|) = \zeta e^{\omega(sN+1)} = \zeta\left(1 + \frac{|z|^2}{4R^2}\right)^{-(sN+1)} \tag{4.4}$$

and a pair potential $-qq' \ln |z - z'|$. The N negative particles stuck at the north pole of the sphere are projected at infinity on the plane.

The formalism for dealing with a plane system with a fugacity dependent on the charge and the position has been set in ref. 5. The n-particle densities can be expressed in terms of Green functions[3] $g_{s_1 s_2}(\mathbf{r}_1, \mathbf{r}_2)$ (here $\mathbf{r}_j$ stands for a position vector in the plane; $z_j = r_j e^{i\varphi_j}$); in the plane, the densities are $2\pi\zeta e^{\omega(r)} g_{ss}(\mathbf{r}, \mathbf{r})$, and the two-body truncated densities are $-s_1 s_2 (2\pi\zeta)^2 e^{\omega(r_1) + \omega(r_2)} |g_{s_1 s_2}(\mathbf{r}_1, \mathbf{r}_2)|^2$. They are the stereographic projections of densities on the sphere, which therefore are

$$\rho_s = (2\pi\zeta)\, e^{-\omega(r)}\, g_{ss}(\mathbf{r}, \mathbf{r}) \tag{4.5a}$$

$$\rho^T_{s_1 s_2}((\theta_1, \varphi_1), (\theta_2, \varphi_2)) = -s_1 s_2 (2\pi\zeta)^2\, e^{-\omega(r_1) - \omega(r_2)}\, |g_{s_1 s_2}(\mathbf{r}_1, \mathbf{r}_2)|^2 \tag{4.5b}$$

where (θ_j, φ_j) is that point on the sphere which is projected at $\mathbf{r}_j$. It is enough to fix one particle at the south pole and to consider only $g_{s_1 s_2}(\mathbf{r}, 0)$.

For the present problem, Eq. (2.18a) of ref. 5, with now $m(\mathbf{r}) = 2\pi\zeta e^{\omega(r)}$, becomes

$$[(2\pi\zeta)^2\, e^{\omega(r)} + A^+ e^{-\omega(r)} A]\; g_{++}(\mathbf{r}, 0) = 2\pi\zeta\delta(\mathbf{r}) \tag{4.6}$$

where

$$A = e^{i\varphi}\left[\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\varphi} - \frac{N}{2}\omega'(r)\right] \tag{4.7a}$$

$$A^+ = e^{-i\varphi}\left[-\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial\varphi} - \frac{N}{2}\omega'(r)\right] \tag{4.7b}$$

Looking for a solution of the form $g_{++}(\mathbf{r}, 0) = g_{++}(r)$ and changing to the variable $s = \sin^2(\theta/2) = e^{\omega(r)}$ and to the function $k(s) = s^{-N/2} g_{++}$, one obtains from (4.6)

$$s(1-s)\frac{d^2k}{ds^2} + [N - (N+1)\,s]\frac{dk}{ds} - (2\pi\zeta R)^2\, k = 0, \qquad s \neq 1 \tag{4.8}$$

[3] In Section 3 of the present paper and in ref. 5, the Green functions have been defined with different irrelevant phase factors, which do not affect the physical quantities.

(4.8) is the hypergeometric differential equation. The two solutions are[11] $F((N+\delta)/2, (N-\delta)/2; N; s)$ and $F((N+\delta)/2, (N-\delta)/2; 1; 1-s)$, where $\delta = (N^2 - 16\pi^2\zeta^2 R^2)^{1/2}$. The second solution behaves like s^{-N+1} as $s \to 0$; it would generate in $\rho^T_{++}(\theta, \pi) = -(2\pi\zeta)^2 s^{N-1} |k(s)|^2$ an unacceptable singularity s^{-N+1} as $s = \sin^2(\theta/2) \to 0$, and it must be discarded. Therefore

$$g_{++}(\mathbf{r}, 0) = C_+ s^{N/2} F\left(\frac{N+\delta}{2}, \frac{N-\delta}{2}; N; s\right) \tag{4.9}$$

From (4.6), near the south pole ($r=0$, $s=1$), $g_{++}(\mathbf{r}, 0)$ behaves like $-\zeta \ln r \sim -(\zeta/2)\ln(1-s)$. The F in (4.9) does have a logarithmic term since

$$\begin{aligned} &F\left(\frac{N+\delta}{2}, \frac{N-\delta}{2}; N; s\right) \\ &\quad \sim \frac{\Gamma(N)}{\Gamma((N+\delta)/2)\,\Gamma((N-\delta)/2)} \\ &\quad\quad \times \left[-2\gamma - \psi\left(\frac{N+\delta}{2}\right) - \psi\left(\frac{N-\delta}{2}\right) - \ln(1-s)\right] \quad \text{as} \quad s \to 1 \end{aligned} \tag{4.10}$$

where Γ is the gamma function, ψ is its logarithmic derivative, and γ is Euler's constant. The appropriate behavior is obtained by choosing

$$C_+ = \frac{\Gamma((N+\delta)/2)\,\Gamma((N-\delta)/2)}{\Gamma(N)} \frac{\zeta}{2} \tag{4.11}$$

Then

$$g_{++}(\mathbf{r}, 0) \sim \frac{\zeta}{2}\left[-2\gamma - \psi\left(\frac{N+\delta}{2}\right) - \psi\left(\frac{N-\delta}{2}\right) - \ln(1-s)\right] \quad \text{as} \quad s \to 1 \tag{4.12}$$

By a similar calculation, from Eq. (2.18b) of ref. 5, one finds

$$g_{--}(\mathbf{r}, 0) = C_- s^{N/2+1} F\left(1 + \frac{N+\delta}{2}, 1 + \frac{N-\delta}{2}; N+2; s\right) \tag{4.13}$$

where

$$C_- = \frac{\Gamma(1 + (N+\delta)/2)\,\Gamma(1 + (N-\delta)/2)}{\Gamma(N+2)} \frac{\zeta}{2} \tag{4.14}$$

such that

$$g_{--}(\mathbf{r},0)\sim\frac{\zeta}{2}\left[-2\gamma-\psi\left(1+\frac{N+\delta}{2}\right)\right.$$
$$\left.-\psi\left(1+\frac{N-\delta}{2}\right)-\ln(1-s)\right]\qquad\text{as}\quad s\to 1\tag{4.15}$$

Finally, from Eq. (2.18c) of ref. 5, one finds

$$g_{-+}(\mathbf{r},0)=\frac{((N+\delta)/2)((N-\delta)/2)}{N}\frac{C_+}{2\pi\zeta R}e^{i\varphi}(1-s^2)^{1/2}s^{(N+1)/2}$$
$$\times F\left(1+\frac{N+\delta}{2},1+\frac{N-\delta}{2};N+1;s\right)\tag{4.16}$$

Using the Green functions (4.9), (4.13), and (4.16) in (4.5b), one finds

$$\rho^T_{++}(\theta,\pi)=-\left[\pi\zeta^2\frac{\Gamma((N+\delta)/2)\,\Gamma((N-\delta)/2)}{\Gamma(N)}\sin^{N-1}\frac{\theta}{2}\right.$$
$$\left.\times F\left(\frac{N+\delta}{2},\frac{N-\delta}{2};N;\sin^2\frac{\theta}{2}\right)\right]^2\tag{4.17a}$$

$$\rho^T_{--}(\theta,\pi)=-\left[\pi\zeta^2\frac{\Gamma(1+(N+\delta)/2)\,\Gamma(1+(N-\delta)/2)}{\Gamma(N+2)}\sin^{N+1}\frac{\theta}{2}\right.$$
$$\left.\times F\left(1+\frac{N+\delta}{2},1+\frac{N-\delta}{2};N+2;\sin^2\frac{\theta}{2}\right)\right]^2\tag{4.17b}$$

$$\rho^T_{-+}(\theta,\pi)=\rho^T_{+-}(\theta,\pi)$$
$$=\left[\frac{\zeta}{2R}\frac{\Gamma(1+(N+\delta)/2)\,\Gamma(1+(N-\delta)/2)}{\Gamma(N+1)}\cos\frac{\theta}{2}\sin^N\frac{\theta}{2}\right.$$
$$\left.\times F\left(1+\frac{N+\delta}{2},1+\frac{N-\delta}{2};N+1;\sin^2\frac{\theta}{2}\right)\right]^2\tag{4.17c}$$

Thus, we retrieve the correlation functions derived in Section 3, Eqs. (3.4b), (3.19), and (3.21).

It can be easily checked that the limit $\zeta\to 0$ reproduces the one-component plasma results of ref. 2 and the limit $N\to 0$ reproduces the two-component plasma results of ref. 7.

The thermodynamic limit giving a plane system can also be studied: $N, R\to\infty$, $\theta\to 0$, with fixed background density $\eta := N/4\pi R^2$, fixed fugacity ζ, and fixed distance $r = 2R\cos(\theta/2)$. By using one of Kummer's relations[11]

$$F(a, b; c; z) = (1-z)^{-b} F\left(c-a, b; c; \frac{z}{z-1}\right) \tag{4.18}$$

one can rewrite (4.17a) as

$$\rho^T_{++}(r) \sim -\left[\eta\alpha \frac{\Gamma(N-\alpha)\,\Gamma(\alpha)}{\Gamma(N)} e^{-\pi\eta r^2/2} N^{\alpha}(\pi\eta r^2)^{-\alpha} \times F\left(\alpha, \alpha; N; 1 - \frac{N}{\pi\eta r^2}\right)\right]^2 \tag{4.19}$$

where $\alpha = \pi\zeta^2/\eta$. In the limit $N \to \infty$, $N^{\alpha}\Gamma(N-\alpha)/\Gamma(N) \to 1$ and

$$(\pi\eta r^2)^{-\alpha} F\left(\alpha, \alpha; N; 1 - \frac{N}{\pi\eta r^2}\right) \to e^{\pi\eta r^2/2}(\pi\eta r^2)^{-1/2}\, W_{1/2-\alpha,\, 0}(\pi\eta r^2) \tag{4.20}$$

where W is a Whittaker confluent hypergeometric function.[11] Thus, in the thermodynamic limit,

$$\rho^T_{++}(r) = -[\eta\Gamma(\alpha+1)(\pi\eta r^2)^{-1/2}\, W_{1/2-\alpha,\, 0}(\pi\eta r^2)]^2 \tag{4.21a}$$

Similar calculations give

$$\rho^T_{--}(r) = -[\eta\alpha\Gamma(\alpha+1)(\pi\eta r^2)^{-1/2}\, W_{-1/2-\alpha,\, 0}(\pi\eta r^2)]^2 \tag{4.21b}$$

$$\rho^T_{-+}(r) = \rho^T_{+-} = [\eta\alpha^{1/2}\Gamma(\alpha+1)(\pi\eta r^2)^{-1/2}\, W_{-\alpha,\, 1/2}(\pi\eta r^2)]^2 \tag{4.21c}$$

The results (4.21a) and (4.21b) had been previously obtained[4] in ref. 10.

5. THERMODYNAMICS

In the present generalized grand-canonical formalism, the basic thermodynamic function is the generalized grand potential Ω obtained from the generalized grand partition function:

$$\Omega = -k_B T \ln \Xi \tag{5.1}$$

[4] The sign of the background charge density has been chosen positive in ref. 10, negative in the present paper. Furthermore, there are sign inconsistencies at the bottom of p. 127 of ref. 10.

From (2.4a), the density of negative particles is

$$\rho_- = \frac{\langle M \rangle}{4\pi R^2} = \frac{1}{8\pi R^2} \zeta \frac{d}{d\zeta} \ln \Xi \tag{5.2}$$

Thus, we can obtain $\ln \Xi$ and Ω by integrating ρ_-/ζ with respect to ζ.

The densities ρ_s given by (4.5a) should be independent of $\mathbf{r}$ and it is convenient to compute them at $\mathbf{r}=0$. However, for point particles, $g_{ss}(0, 0)$ is a divergent quantity, as already noticed in simpler cases.[5,7] For obtaining finite densities, we assume the particles to be small hard disks of diameter σ rather than point particles. This regularization[5] amounts to replacing $g_{ss}(0, 0)$ by $g_{ss}(\sigma, 0)$. Since σ is small, we can use (4.12) and (4.15), with $1-s=\sigma^2/4R^2$, in (4.5a), which gives the densities

$$\rho_+ = \pi\zeta^2 \left[-2\gamma - \psi\left(\frac{N+\delta}{2}\right) - \psi\left(\frac{N-\delta}{2}\right) + 2 \ln \frac{2R}{\sigma} \right] \tag{5.3a}$$

$$\rho_- = \pi\zeta^2 \left[-2\gamma - \psi\left(1+\frac{N+\delta}{2}\right) - \psi\left(1+\frac{N-\delta}{2}\right) + 2 \ln \frac{2R}{\sigma} \right] \tag{5.3b}$$

Using $\psi(1+x) - \psi(x) = 1/x$, one easily checks that $\rho_+ - \rho_-$ is equal to the background density $\eta = N/4\pi R^2$.

From now on, we only consider the case of a large system, using large-R expansions for fixed values of the fugacity ζ and the background density $\eta = N/4\pi R^2$. As R becomes large, an expansion of the ψ functions gives

$$\begin{aligned} \rho_- &= \rho_+ - \eta \\ &= \eta\alpha[-2\gamma - \psi(1+\alpha) - \ln \pi\eta\sigma^2] \\ &\quad + \frac{\alpha}{4\pi R^2}\left[\frac{1}{2} + \alpha - \alpha^2\psi'(\alpha)\right] + O\left(\frac{1}{R^4}\right) \end{aligned} \tag{5.4}$$

where, again, $\alpha = \pi\zeta^2/\eta$.

For $\zeta = 0$, Ξ should become the partition function of the one-component plasma[2] (the thermal de Broglie wavelength has been taken as unity)[5]

$$Z = e^{N^2/2}(2\pi R)^N \prod_{p=0}^{N-1} \frac{p!\,(N-p-1)!}{N!} \tag{5.5}$$

[5] Our definition of Z includes the usual $1/N!$ factor which is not present in the "excess partition function" defined in ref. 2.

as confirmed in the Appendix. The corresponding free energy $\Omega(0) = -\beta^{-1} \ln Z$ can be expanded[6] to give

$$\frac{1}{k_B T} \Omega(0) = 4\pi R^2 \frac{\eta}{2} \ln \frac{\eta}{2\pi^2} + \frac{1}{3} \ln [(4\pi\eta)^{1/2} R] + \frac{1}{12} - 2\zeta'(-1) + o(1) \quad (5.6)$$

where ζ' is the derivative of Riemann's zeta function.

Using (5.4) in (5.2) and integrating from $\zeta = 0$ gives the large-R expansion

$$\frac{1}{k_B T} \Omega = \frac{1}{k_B T} \Omega(0) + 4\pi R^2 \eta [\alpha(2\gamma + \ln \pi\eta\sigma^2) + \ln \Gamma(1+\alpha)] \\ - \frac{\alpha^2}{2} - \frac{\alpha}{2} + \int_0^\alpha x^2 \psi'(x)\, dx + o(1) \quad (5.7)$$

The quantity $(k_B T)^{-1} \Omega$ does exhibit the expected universal finite-size correction[6] $(1/3) \ln R$. That correction comes entirely from $\Omega(0)$.

APPENDIX

When $\zeta = 0$ and $a = b = 1$,

$$\Xi_2(a, b) = Z_2 \quad (A.1)$$

where Z_2 denotes the canonical partition function of the one-component plasma on a sphere. Using (2.12), we thus have

$$Z_2 = A_{N2} \prod_k \lambda_k \quad (A.2)$$

where the product is over all eigenvalues of the operator $\mathbb{X}$. The operator $\mathbb{X}$ again acts on vectors (2.13), except that the component $f(\theta, \varphi)$ is no longer present. The eigenvectors are therefore of the form

$$\psi_k = \begin{bmatrix} a_1^{(k)} \\ \vdots \\ a_N^{(k)} \\ g^{(k)}(\theta, \varphi) \end{bmatrix} \quad (A.3)$$

and from (2.15) the eigenvalues and eigenvectors are specified by the equations

$$\lambda_k a_j^{(k)} = i \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi\, [\sin(\theta/2)]^{N-1} [\cot(\theta/2)\, e^{i\varphi}]^{j-1} g^{(k)}(\theta, \varphi) \quad (A.4)$$

$(j = 1,..., N)$ and

$$\lambda_k g^{(k)}(\theta, \varphi) = g^{(k)}(\theta, \varphi) + i \sum_{j=1}^{N} a_j^{(k)}[\cot(\theta/2)\, e^{-i\varphi}]^{j-1} [\sin(\theta/2)]^{N-1} \quad (A.5)$$

We look for solutions of the above equations of the form

$$g^{(k)}(\theta, \varphi) = e^{ik\varphi} g_k(\theta), \qquad k \in \mathbb{Z} \quad (A.6)$$

Substituting (A.6) in (A.4) gives

$$a_j^{(k)} = 0, \qquad k \neq 1 - j \quad (A.7)$$

$$\lambda_k a_j^{(k)} = 2\pi i \int_0^\pi d\theta \sin\theta\, [\sin(\theta/2)]^{N-1} \times [\cot(\theta/2)]^{j-1} g_k(\theta), \qquad k = 1 - j \quad (A.8)$$

and use of these equations gives from (A.5)

$$(\lambda_k - 1)\, g_k(\theta) = 0, \qquad k \notin \{0,..., -(N-1)\} \quad (A.9a)$$

and

$$(1 - \lambda_k)\, \lambda_k = 2\pi \int_0^\pi d\theta \sin\theta \left(\sin\frac{\theta}{2}\right)^{2(N-1)} \left(\cot\frac{\theta}{2}\right)^{-2k} = 4\pi \frac{\Gamma(N+k)\, \Gamma(-k+1)}{\Gamma(N+1)}, \qquad k = 0, -1,..., -(N-1) \quad (A.9b)$$

Equation (A.9a) gives the eigenvalues $\lambda_k = 1$ for all $k \notin \{0,..., -(N-1)\}$, while (A.9b) gives a pair of eigenvalues λ_k^+ and λ_k^- for each $k = 0,..., -(N-1)$ such that their product $\lambda_k^+ \lambda_k^-$ is equal to the r.h.s. of (A.9b).

Thus, writing $p = -k$, we have

$$Z_2 = A_{N2} \prod_{p=0}^{N-1} \left[4\pi \frac{\Gamma(N-p)\, \Gamma(p+1)}{\Gamma(N+1)} \right] \quad (A.10)$$

which is precisely the result (5.5), derived from the Vandermonde formalism of ref. 2.

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